How Far Up Am I?
The Mathematics of Stadium Seating

Rebecca McGraw, David Romero, and Robert Krueger

When students can connect mathematical ideas, their understanding is deeper and more lasting. They can see mathematical connections in the rich interplay among mathematical topics, in contexts that relate mathematics to other subjects, and in their own interests and experience. Through instruction that emphasizes the interrelatedness of mathematical ideas, students not only learn mathematics, they also learn about the utility of mathematics.

—National Council of Teachers of Mathematics 2000, p. 64

When we make mathematics meaningful for our students through connections to the real world, we deepen their understandings and help them see the power and usefulness of the subject. Investigating mathematical relationships in real-world contexts, students can use the tables, graphs, and equations of algebra as tools for interpreting and predicting. Student knowledge of mathematics is enhanced by a curriculum that includes multiple opportunities to collect and analyze data, work with multiple representations, and compare and discuss results (Boaler 1998; Senk and Thompson 2003). In this article, we describe a lesson in which students develop understanding of linear functions as they investigate the height of football stadium bleachers.

THE STADIUM SEATING PROBLEM

We introduce this lesson to algebra students by showing them a diagram of the school's football stadium bleachers (see **fig. 1**). After a discussion of how this two-dimensional sketch represents the bleachers, we distribute a recording sheet to each student (see **fig. 2**) and pose the following questions:

- If you were sitting on a bleacher in the first row, how high above the ground would you be?
- If you were sitting on a bleacher in the fifth row, how high above the ground would you be?
- Can you find an equation that would allow you to determine how high a person is above the ground for any given bleacher row?

Because the two-dimensional sketch includes only one measurement, the height of the walkway above the ground, students must determine what data they need to collect in order to solve the problem. Groups of three or four students plan what data to collect before going to the football field. Each student is required to copy the two-dimensional sketch of the bleachers onto the back of the handout (**fig. 2**), and each group is given a yardstick with which to take measurements. Once outside, students can record their measurements directly onto their sketches. Our experience working with beginning algebra students suggests that students need approximately fifteen minutes to collect data at the football field. If taking students outside is difficult or if the school does not have a football stadium, this activity can be done just as easily using the school's basketball bleachers or other school risers.

Figure 3 shows the stadium bleacher diagram with approximate measurements taken from the high school in which we have used the activity. One equation that would accurately represent the relationship between the bleacher number (x) and the height of the bleacher above the ground (y) would be $y = 10x + 47$; another correct equation would be $y = 57 + 10(x - 1)$. We have found that beginning algebra students' ability to graph the bleacher data and find equations that accurately represent the relationship between bleacher number and bleacher height varies greatly. However, student work on this problem provides us with a starting point for discussing a variety of issues related to drawing graphs and writing equations. **Figure 4** shows several examples of the graphs and equations students produce.

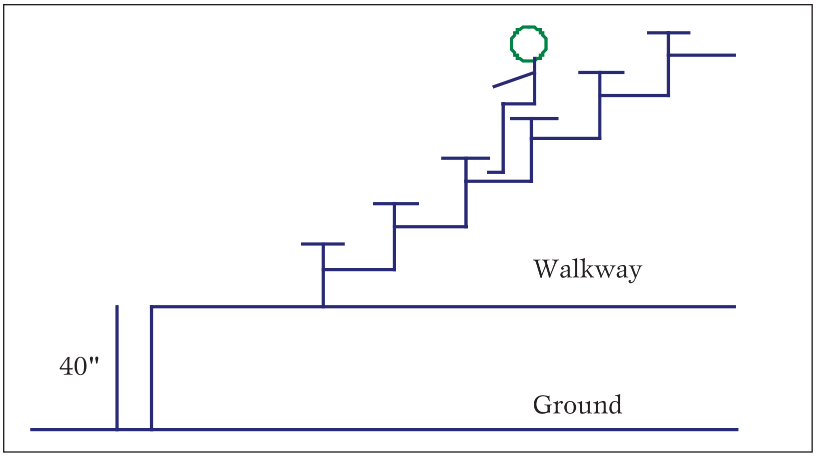
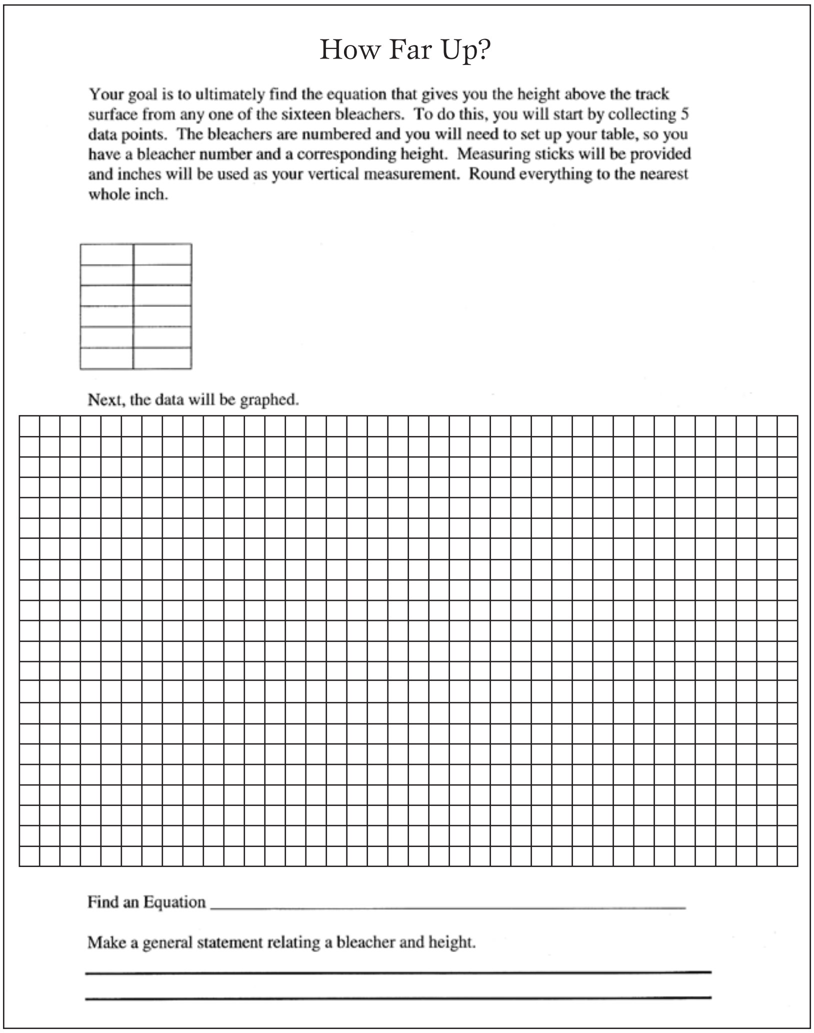


Fig. 1 Diagram representing the football stadium bleachers


How Far Up?

Your goal is to ultimately find the equation that gives you the height above the track surface from any one of the sixteen bleachers. To do this, you will start by collecting 5 data points. The bleachers are numbered and you will need to set up your table, so you have a bleacher number and a corresponding height. Measuring sticks will be provided and inches will be used as your vertical measurement. Round everything to the nearest whole inch.

Next, the data will be graphed.

Find an Equation ______________________

Make a general statement relating a bleacher and height.

____

____

Fig. 2 Student handout for the Stadium Seating problem

BEGINNING ALGEBRA STUDENTS' TABLES, GRAPHS, AND EQUATIONS

The "How Far Up?" worksheet (see **fig. 2**) is organized so that students begin by collecting data points and constructing tables that relate bleacher number to height above the ground. Data in the tables reflect student assumptions about the growth in height of the bleachers (see **fig. 4**). For example, David and Melinda both correctly measured the height of the first bleacher above the walkway (17 inches) and calculated the overall height of the first bleacher to be 57 inches above the ground. David, however, appears to believe that each subsequent seat will be 17 inches higher above the ground than the previous seat—a reasonable but erroneous conclusion that is then accurately represented in his equation. Melinda, who chose to include every third bleacher in her table, correctly found the change in bleacher height to be 10 inches per bleacher. Note, however, that this relationship is *not* represented in her equation. We would need to investigate her thinking further in order to understand how she derived her equation.

Analyzing David's and Melinda's graphs, we note that David's choice of scale for the x- and y-axes is reasonable and that he was able to graph the data in his table accurately. Although David had to estimate the locations of some data points, we can see the linear relationship between bleacher number and height by looking at his graph. Melinda seems to have tried to create x- and y-scales for her graph in such a way that estimating point locations would not be necessary, since each point would fall on an intersection of the grid lines. Had she numbered the y-axes accurately (she skipped 177) and used evenly spaced intervals, she too would have had a reasonable and accurate graph of the data. In fact, with each point falling on an intersection of gridlines, the rate of change of the bleacher height (30 inches per 3 bleachers, or 10 inches per bleacher) would have been particularly easy to see in her graph.

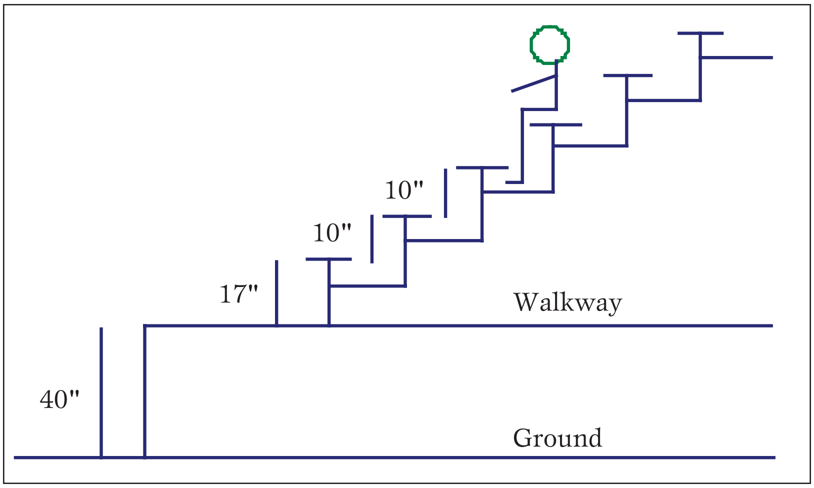


Fig. 3 Diagram of football stadium bleachers with measurements

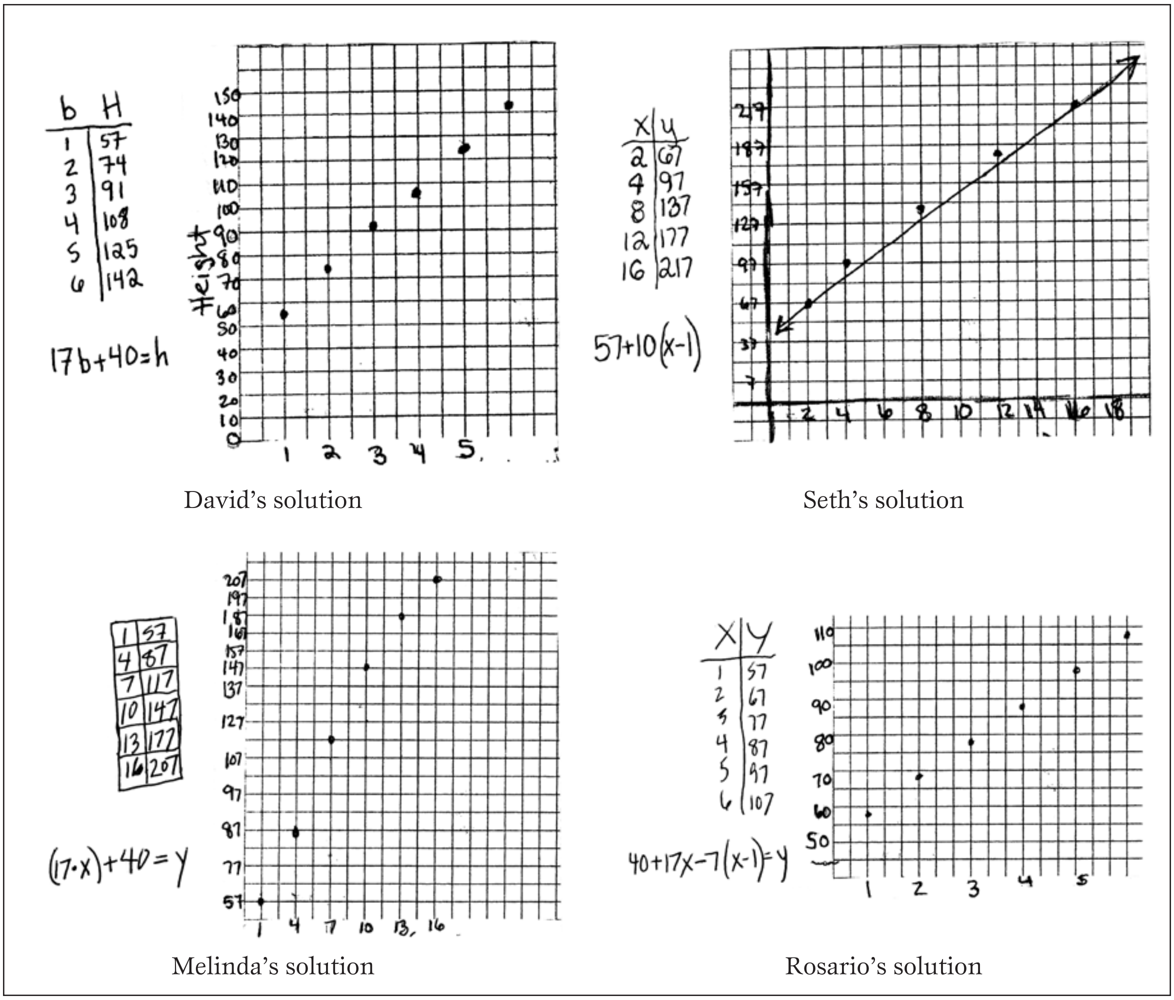


Fig. 4 Examples of student solutions to the Stadium Seating problem

Both Seth and Rosario wrote correct, though quite distinct, algebraic representations of the bleacher-height relationship. Seth took the first bleacher seat height (57 inches) as a constant and used $(x - 1)$ to adjust for the fact that the 10-inch-per-bleacher rate of change begins with the second bleacher rather than the first. Rosario began with the 40-inch walkway height, added $17x$ for the additional height of the first bleacher, and subtracted 7 inches from each additional bleacher to adjust for the change in the rate of change. While Rosario's table and graph are consistent with her equation, Seth's are not consistent with his algebraic expression. All but one of the points in his table and graph are consistent with the function $57 + 10x = y$, indicating that perhaps the inclusion of $(x - 1)$ rather than just x in his equation was an afterthought. Looking across the four students' solutions, we notice that only David numbered the y-axis consistently beginning with zero, only Seth wrote an expression rather than an equation, and only Seth drew a line through the points on the graph. It is not clear from looking at the students' work whether the other three students considered drawing a line through their points; however, differences in the graphs of discrete and continuous data would be an interesting topic for whole-class discussion.

Allowing students to determine the values in their tables, the scales of their graphs, and the forms of their equations is important for many reasons, not least of which is that students' ways of thinking become more visible to us. We can then engage students in comparing solutions and help them identify any misconceptions or inconsistencies in their representations. In the following section, we use a vignette of whole-class discussion to illustrate how we encourage students to analyze one another's work and make connections across representations.

WHOLE-CLASS DISCUSSION

Whole-class discussion of students' solutions to the Stadium Seating problem is a central component of the lesson. When students share their thinking and analyze one another's tables, graphs, and equations, they further their understanding of how one representation is connected to another. They also develop important mathematical habits, including communicating ideas verbally, reasoning about mathematical relationships, and providing justification for solutions and methods. In the following example discussion, notice how Mr. Ruiz presses students to clarify and justify their thinking and how he helps students make connections among the equations, the tables of values, and the real-world context of the stadium seats.

Classroom Vignette

*Mr. Ruiz begins the discussion by asking several students to come to the whiteboard and write the equations they found to describe the relationship between the bleacher row and the height of the bleacher. To the side of the equations (shown below), Mr. Ruiz has sketched a diagram of the stadium seats (see **fig. 1**).*

1. $47 + 10x = y$	2. $57 + 10x = y$
3. $(17x) + 10 = y$	4. $40 + 17b = h$

Mr Ruiz: I'm not worried about if an equation is right or wrong right now. I want to know how you got the numbers.
Jessie: Well, for the first equation, if it's the first bleacher, then it has to be 57 above the ground because you add 40 plus 17, and the equation matches because it would be 47 plus 10.

[*Mr. Ruiz creates a table showing "seat" and "height" as column headings beside the first equation. Mr. Ruiz writes "1" under "seat" and "57" under "height."*]

Mr. Ruiz: Do you agree with Jessie that if we put 1 into the equation for x, then we will get 57 as our answer? [*Most students murmur "yes" as Mr. Ruiz writes* $47 + 10(1) = y$, $47 + 10 = y$, *and* $57 = y$ *underneath the first equation.*]

Mr. Ruiz: What about seat 2? [*Mr. Ruiz enters* 2 *into the table and leads the class in substituting* 2 *into the equation to get* 67.] So, is 67 what we should get?
Marisa: Yes, because you just do 10 times 2 plus 47.
Mr. Ruiz: Yes, you are right, that's what the equation gives us. But do 57 and 67 make sense in terms of the stadium seats? [*Mr. Ruiz moves so that he is standing next to the diagram on the whiteboard of the stadium seats.*]
Marisa: First you add 17, but after that you add 10, so it has to go up by 10 each time.
Javier: Yeah. It has to be 77, 87, 97. It just keeps going.
Mr. Ruiz [*to the class*]**:** What measurements did you come up with when you were outside? How high was the first seat above the walkway? Marisa says first she added 17. [*Speaking to Marisa:*] Do you mean 17 inches from the walkway to the first seat? [*Marisa nods.*] What did other people find?
Jeff: Our group got 17 for each seat. When you go from one seat to the next, I mean, we got 17 each time. We got $40 + 17b$.
Marisa: But it is only 17 the first time. It's less for the rest because the seats stick up.

[*At this point, Mr. Ruiz uses the diagram of the stadium seats to aid students in analyzing the difference* *between repeatedly adding* 17 *and repeatedly adding* 10. *Consensus develops among the students that the vertical distance between seats is* 10 *inches, not* 17, *and Mr. Ruiz adds these measurements to the diagram (see* ***fig. 3****)*].

Mr. Ruiz: So we can fill in the table for the first equation now, right? If we keep adding 10, we'll get 77, then 87, then 97 for seats 3, 4, and 5. Does that match our diagram? [*Students nod.*] Okay. How about the second equation?
Nick: Well, the equation's not right.
Mr. Ruiz: How do you know?
Nick: The numbers will be off.
Mr. Ruiz: The numbers in the table? What will those numbers be?
Nick: Well, for the first bleacher you'd have 67.

[*As Nick speaks, Mr. Ruiz begins to draw a table underneath the second equation. He enters* 1 *and* 67 *as the first values.*]

Mr. Ruiz [*to the class*]: Is that what the equation gives? Viviana, what will the height be for seat 2 for this equation?
Viviana: Umm. 77?
Mr. Ruiz: How did you get that?
Viviana: Well, 20 plus 57.
Mr. Ruiz: Okay. And the next seat?
Viviana: 87, then 97. Ten each time, like the other one. [*Mr. Ruiz continues to fill in the table according to Viviana's directions.*]
Nick: But the numbers are still wrong—like they're 10 too much each time.
Mr. Ruiz: So we need to have 10 less for each row in our table. How could we change the equation just slightly so that we would get 10 less each time?
Javier: You could just change the 57 to 47. Then it would be the same as the first one.
Mr. Ruiz: Yes, that would work. That would be a good idea. [*pause*] Does anyone see another way? [*long pause*] How about this? [*Mr. Ruiz writes* $57 + 10(x - 1) = y$ *above the table and underneath the second equation. Then he makes a third column in the table.*]
Mr. Ruiz: Try to find the heights using this equation. Does this equation work? [*Students do the calculations and agree that the equation works.*]
Mr. Ruiz: Didn't I see this equation on the papers of one group? Who was that? [*Seth's group raises their hands.*] Did your table match this table?
Seth: Well, for seat 2 we got 67, but for seat 4 we got 97. That's not right.
Mr. Ruiz: Why do you say it isn't right?
Anna: It should be 87, not 97. You can tell by looking at the seats, it's 57 plus two 10s, not three.
Seth: Our graph is wrong, too.
Mr. Ruiz: That's okay. Do you understand it now? Okay. Now equation numbers 3 and 4 are different from the first two, and they will give us different tables of values and graphs as well. Please take a couple of minutes and figure out what the values in each table will be for seats 1, 2, 3, and 4. Go ahead and do this whether or not one of these is your group's equation.

[*Mr. Ruiz gives students time to construct the tables while he walks around and checks their understanding of substitution and order of operations. Then he puts the tables and values on the whiteboard.*]

Mr. Ruiz: Okay. When we were outside, you had to collect data and then find an equation to represent the data. I'm going to ask you to do the opposite now. You have two equations, $(17x) + 10 = y$ and $40 + 17b = h$, and you have tables of values that go with those equations. Your job is to work with your group to draw a picture of a series of stadium seats for which $(17x) + 10 = y$ would be the correct equation. Of course the measurements for the heights will be different, but you can use the equation and table to figure out what those measurements would be. Once you have a diagram, I will check it, and then you can move on and do $40 + 17b = h$. Any questions?

Mr. Ruiz began the whole-class discussion by working backward—from the equations to tables of values to the diagram of the stadium seats. This process of analysis is one that students can use in the future to assess the correctness of their own equations. As the discussion shifts to the second equation, students quickly realize that the values in the table do not match the diagram and that the equation needs to be changed. Mr. Ruiz then helps students see that $47 + 10x = y$ is not the only equation that accurately represents the data and that $57 + 10(x - 1) = y$ makes sense in terms of both the diagram and the table of values. For equations 3 and 4, Mr. Ruiz turns responsibility for constructing tables over to the students. Moving beyond simply agreeing that the equations do not represent the stadium seats, Mr. Ruiz capitalizes on students' growing understanding of the relationship between the diagram and the equations by asking them to create diagrams for which the equations would be correct. Throughout the discussion, Mr. Ruiz uses questions such as "Why do you say it isn't right?" "Does anyone see another way?" and "How do you know?" to encourage students to justify and expand their thinking.

The equations students write represent their ways of thinking about the relationship between bleacher number and height. Rosario, who wrote $40 + 17x - 7(x - 1) = y$, explained her equation this way:

> I took the walkway height and added $17x$ because that was the height for the first bleacher, but that was too much for each of the other bleachers, so I had to take 7 away for each bleacher after the first one.

When students share this kind of thinking with their classmates, they begin to see that equations that look very different from one another can accurately represent the same relationship. Additional evidence is provided by testing values and realizing that different equations produce the same tables. Once students have investigated various equations and determined which ones "work," the teacher can help students see how every equation can be simplified to $47 + 10x = y$. We are careful, however, not to overemphasize the simplified form.

Important connections can be made from student work on the Stadium Seating problem to concepts such as rate of change and y-intercept. Asking a question such as "How would the table (or graph or equation) change if the change in height of the seats was 15 inches instead of 10 inches?" serves to focus attention on the rate of change. On their graphs, students can see that a change of 1 bleacher in the horizontal direction corresponds to a change of 10 inches in the vertical direction—the rate of change of 10 inches per bleacher. With respect to the y-intercept, students might consider how changes in the height of the walkway would affect the table, graph, and equation. Students should also be encouraged to use their equations to predict the heights of the seats, should more bleachers be added. For example, the teacher might ask, "How far up would I be if there were 25 rows of bleachers and I was sitting in the 25th row?" "What if there were 100 rows?" Conversely, one might ask, "If I am 127 inches above the ground, what bleacher number must I be sitting on?" Questions such as this one give students opportunities to practice solving an equation for x—especially when large numbers are given for y.

Teachers can extend student understanding developed during the stadium seating lesson in many ways. For instance, students might take staircase measurements at school, at home, or in their community and create diagrams, tables, graphs, and equations to represent the relationship between the stair number and height above the ground or floor. They could also measure and compare the relative steepness, or slope, of various sets of risers (if available) or staircases.

CONCLUSION

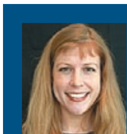
The Stadium Seating problem provides students with a real-world context through which they can investigate a linear function and make connections across multiple representations. This lesson helps algebra students see mathematics as a meaningful way to describe relationships between objects, and it provides a foundation upon which students can build their knowledge of concepts such as rate of change, and their ability to create graphs and write equations. While the Stadium Seating problem is recommended for use with beginning algebra students, we can imagine utilizing investigations that are similarly structured when students study quadratic, exponential, and trigonometric functions.

REFERENCES

Boaler, Jo. "Open and Closed Mathematics: Student Experiences and Understandings." *Journal for Research in Mathematics Education* 29 (January 1998): 41–62.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, VA: NCTM, 2000.

Senk, Sharon L., and Denisse R. Thompson, eds. *Standards-Based School Mathematics Curricula: What Are They? What Do Students Learn?* Mahwah, NJ: Lawrence Erlbaum Associates, 2003. ∞



REBECCA McGRAW, rmcgraw@math.arizona.edu, teaches mathematics content and methods courses at the University of Arizona, Tucson, AZ 85721, and her interests include classroom discourse and student reasoning. DAVID ROMERO, D.C.Romero@maranausd.org, teaches algebra at Marana Mountain View High School in Tucson, AZ 85742. He is interested in alternative strategies for teaching mathematics. ROBERT KRUEGER, BobK@susd12.org, teaches mathematics at Desert View High School in Tucson, AZ 85706. His interests include helping students understand the role of mathematics in the real world.