John F. Mahoney

Benjamin Banneker's Mathematical Puzzles

Benjamin Banneker was a scientist, scholar, astronomer, and mathematical wizard. An African American, he was born a free man in Maryland in 1731. Banneker was known and respected throughout America and Europe for his mathematics skills. Using only a pocket watch as a model, he built the first wooden striking clock in America. It kept accurate time for fifty-three years. President Jefferson had such high regard for Banneker that he chose him to assist in developing plans for laying out the streets of Washington, D.C. From 1792 to 1797, Banneker published a *Farmer's Almanac,* which contained his calculations of the positions of the sun, moon, and planets. These calculations enabled him to predict eclipses and weather conditions. Banneker died in 1806 on his Maryland farm.

As a self-taught mathematician, Banneker kept descriptions of mathematical puzzles in his journals. Silvio Bedini's (1999) excellent biography of Banneker contains a selection of these puzzles. Although these puzzles are of historical interest, some of them may also be used to demonstrate the ways that available technology can be used to efficiently explore and solve problems. This article uses a TI-89 calculator to explore four of those puzzles.

Banneker built the first wooden striking clock in America

PUZZLE 1

A gentleman Sent his Servant with £100 to buy 100 Cattle, with orders to give £5 for each Bullock, 20 Shillings for cows, and one Shilling for each Sheep, the question is to know what number of each sort he brought to his master. (Bedini 1999, p. 326)

This puzzle is a version of a well-known problem, and its solution is an example of a system of Diophantine equations. Diophantine equations are equations for which only integers are considered for solutions. In the eighteenth century, the word *cattle* was used to describe domesticated animals, including sheep, goats, cows, and bulls; and a shilling was 1/20 of a pound (£). If we let b equal the number of bullocks (young bulls), c equal the number of cows, and s equal the number of sheep, then we quickly get the two equations shown in **figure 1:**

$$b + c + s = 100$$

$$5b + c + \frac{s}{20} = 100$$

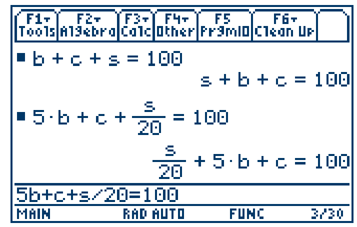

Fig. 1
A system of Diophantine equations

The calculator can solve simultaneous equations. The syntax for the command is solve (b + c + s = 100 and s/20 + 5 • b + c = 100, {b,c,s}), as shown in **figure 2.**

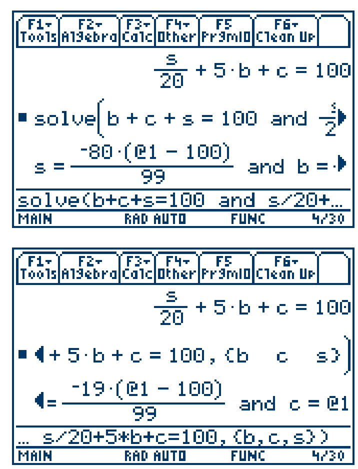

Fig. 2
Solving the system of equations simultaneously

How do we interpret the result,

$$s = \frac{-80 \cdot (@1-100)}{99} \text{ and } b = \frac{-19 \cdot (@1-100)}{99}$$

and c = @1?

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We should remember that we have asked the calculator to do a difficult job, that is, to solve two equations with three unknowns. This calculator uses the symbol @1 to represent an arbitrary value for *c*. The value @1 also determines the values of each of the other two variables, *s* and *b*. What values of @1 result in meaningful solutions to our problem? Our solutions must be integers, since we cannot have two-thirds of a cow. We can use the calculator's table feature to represent solutions. If we let *x* equal @1, then we can represent the number of bullocks with the equation

$$y1(x) = \frac{-19 \cdot (x - 100)}{99},$$

the number of cows with the equation $y2(x) = x$, and the number of sheep with the equation

$$y3(x) = \frac{-80 \cdot (x - 100)}{99},$$

as shown in **figure 3.**

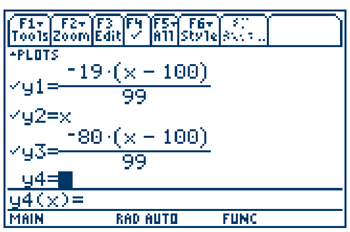

Fig. 3
Equations that represent the number of bullocks, cows, and sheep

We can next construct a table, shown in **figure 4,** with integral values for *x* so that we can examine some of the possibilities.

We are lucky that the very first line of the table, when $x = 1$, gives us a solution with integral values: {19, 1, 80}. If we scroll down the table, though, we can see that no other set of solutions appears to be composed entirely of integers. On further reflection, we should have realized that this result is not surprising, because our equations for bullocks and sheep both have a denominator of 99. The only way for them to have integral values is for their numerators to be factors of 99 and in particular, for $(x - 100)$ to be a factor of 99. The only value of *x* where $0 \le x \le 100$ that makes $(x - 100)$ a factor of 99 is $x = 1$. Therefore, the servant should buy nineteen bullocks at £5 each, one cow at 20 shillings, and eighty sheep at 1 shilling each. This solution is the same one that Banneker recorded in his journal.

We are lucky that the first line of the table gives a solution with integral values

PUZZLE 2

Suppose a ladder 60 feet long be placed in a Street so as to reach a window on one Side 37 feet high, and without moving it at bottom, will reach another window on the other side of the Street which is 23 feet high, requiring the breadth of the Street. (Bedini 1999, p. 326)

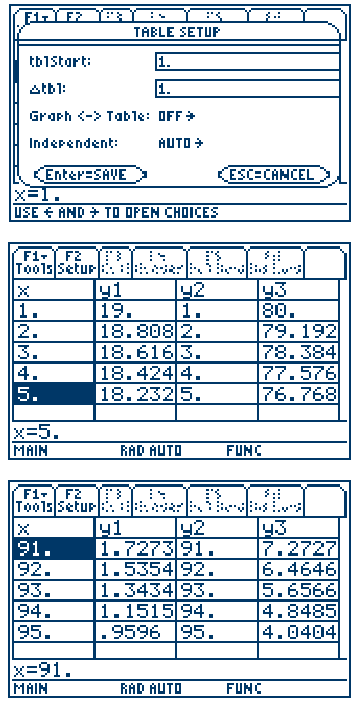

Fig. 4
Constructing a table with integral powers of *x*

The diagram in **figure 5** helps us see that we can find the width, or breadth, of the street by applying the Pythagorean theorem twice to find the distances from the bottom of the ladder to the sides of the street and then adding these distances together, as shown in **figure 6.**

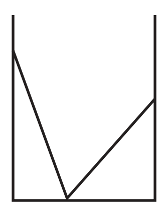

Fig. 5
Diagram of ladder placement for puzzle 2

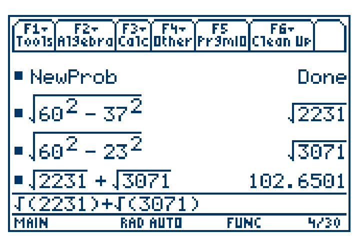

Fig. 6
Applying the Pythagorean theorem to puzzle 2

The width of the street is therefore approximately 102.65 feet. Banneker did not record a solution to this problem in his journal.

PUZZLE 3

Divide 60 into four Such parts, that the first being increased by 4, the Second decreased by 4, the third multiplyed by 4, the fourth part divided by 4, that the Sum, the difference, the product, and the Quotient shall be one and the Same Number. (Bedini 1999, p. 324)

What are the important facts that the cooper must use to construct this vessel?

Several ways exist to solve this problem with the TI-89. One involves writing five equations and solving them simultaneously, as shown **in figure 7.** The last command is too long to display in one screen shot. It is solve (a + b + c + d = 60 and a + 4 = x and b – 4 = x and c • 4 = x and d/4 = x, {x, a, b, c, d})

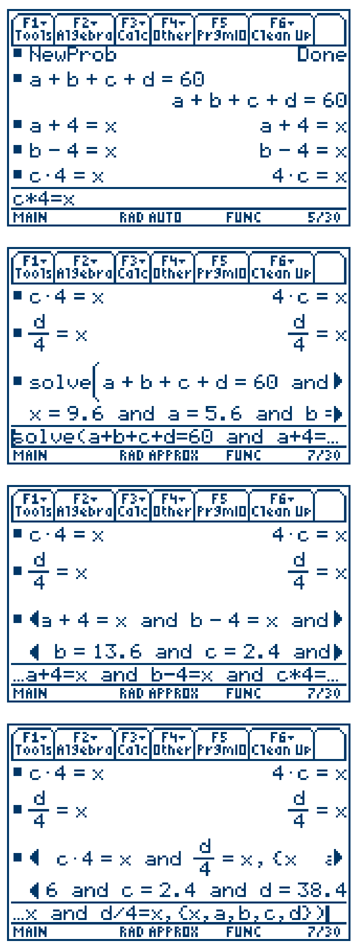

Fig. 7
Solving five equations simultaneously for puzzle 3

The four parts of 60 are therefore 5.6, 13.6, 2.4, and 38.4; and the "same number" is 9.6.

Another way of solving the problem is to just substitute expressions for each of the parts of 60 in terms of the same number, *x,* to get the equation $(x - 4) + (x + 4) + x/4 + 4 \cdot x = 60$, as shown in

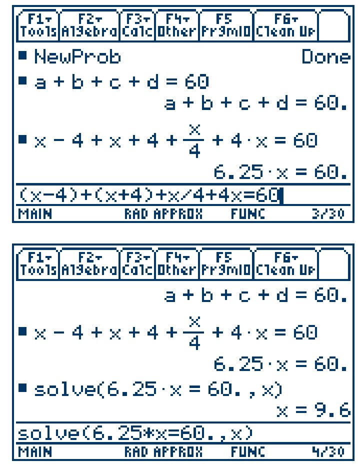

Fig. 8
Substituting expressions in terms of *x* for puzzle 3

figure 8, and then to calculate the four parts on the basis of this value of *x.*

PUZZLE 4

A cooper and vintner sat down for a talk,
Both being so groggy that neither could walk;
Says cooper to vintner, "I'm the first of my trade,
There's no kind of vessel but what I have made,
And of any shape, sir, just what you will,
And of any size, sir, from a tun to a gill."
"Then," says the vintner, "you're the man for me.
Make me a vessel, if we can agree.
The top and the bottom diameter define,
To bear that proportion as fifteen to nine,
Thirty-five inches are just what I crave,
No more and no less in the depth will I have;
Just thirty-nine gallons this vessel must hold,
Then I will reward you with silver or gold,—
Give me your promise, my honest old friend."
"I'll make it tomorrow, that you may depend!"
So, the next day, the cooper, his work to discharge,
Soon, made the new vessel, but made it too large:
He took out some staves, which made it too small,
And then cursed the vessel, the vintner, and all.
He beat on his breast, "By the powers" he swore
He never would work at his trade any more.
Now, my worthy friend, find out if you can,
The vessel's dimensions, and comfort the man!
(Bedini 1999, pp. 326–27)

What are the important facts that the cooper must use to construct this vessel? The height must be 35 inches, it must contain 39 gallons, and the ratio of the diameter of the top to that of the bottom must be 15 to 9.

The cooper could consider making the vessel in a variety of shapes. One of these might be shaped like a large paper cup, as shown in **figure 9.**

Banneker lived before the current system of weights and measures was adopted by Congress;

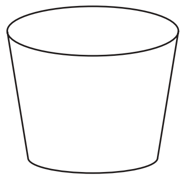

Fig. 9
One shape the cooper could use for the vessel

therefore, the reference to gallons would have referred to the British imperial gallon, which is approximately 20 percent larger than the current American gallon. Our calculator can help us solve this problem. It has a built-in unit menu that allows us to convert the volume of 39 imperial gallons to cubic inches, as shown in **figure 10.**

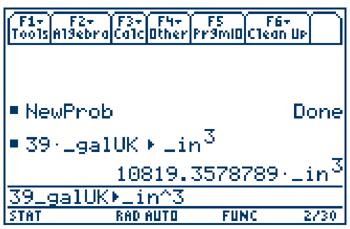

Fig. 10
Converting the volume of 39 imperial gallons to cubic inches

Our paper-cup-shaped figure is actually part of a cone and is called a *frustrum*. We can find an expression for its volume by using the formula for the volume of a cone. We use the diagram in **figure 11** to represent the cross section of the vessel.

Since the diameters of the top and bottom are in the ratio of 15 to 9, so are their radii; therefore, we can call the top 15*k* and the bottom 9*k,* where *k* is a constant. If we let *h* stand for the height of the bottom part of the cone, then we can express the volume of the vessel as the difference in volumes of the two similar cones. The formula for the volume of a right circular cone is

$$V = \frac{1}{3}\pi r2h,$$

where *r* is the radius and *h* is the height of the cone. First, though, we need to find the value of *h.* Because the triangles in the diagram are similar, their sides are in the ratio of 15 to 9. Therefore, we can form an equation relating that ratio, 35, and *h,*

$$\frac{15}{9} = \frac{35+h}{h}$$

and solve it for *h*, as shown in **figure 12.** Then we store the value of *h.* Next, we can find expressions in terms of *k* for the volumes of the two cones,

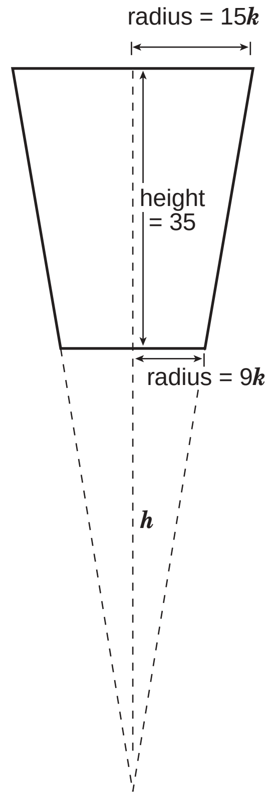

Fig. 11
A representation of the cross section of the vessel

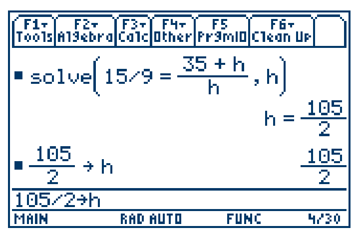

Fig. 12
Solving for *h*

shown in **figure 13,** and an expression for the difference of their volumes, shown in **figure 14,** which is the volume of our vessel.

We previously calculated that the volume of the vessel needed to be 10,819.4 cubic inches for it to hold exactly thirty-nine gallons. We can use that fact to find the value of *k,* as shown in **figure 15.**

The positive value for *k,* 0.8181503, gives us the radii for the top and bottom of the vessel. See **figure 16.**

The cooper could have created a barrel with a bottom diameter of 14.72671 inches, a top one of 24.54451 inches, and a height of 35 inches, so that the barrel would have held exactly 39 gallons. Although Banneker did not provide a solution to

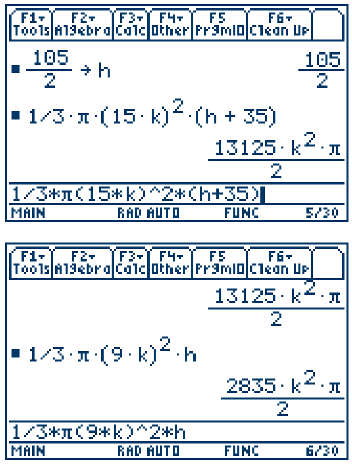

Fig. 13
Expressions in terms of *k* for the volumes of the two cones.

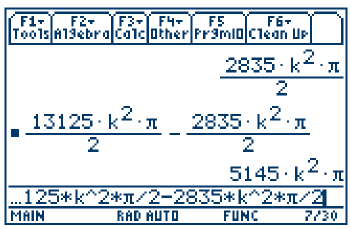

Fig. 14
Expressions in terms of *k* for the difference of the two volumes

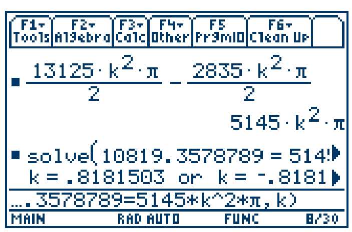

Fig. 15
Finding the value of *k*

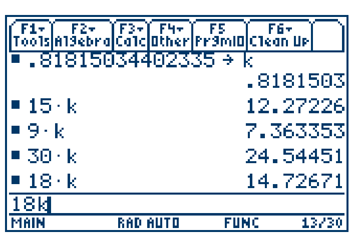

Fig. 16
Using the value of *k* to find the radii for the top and bottom of the vessel

the problem, Bedini (1999, p. 343) noted that in the nineteenth century "Benjamin Hallowell of Alexandria, Virginia was reported to have proposed as his solution 24.745 inches and 14.8476 inches for the diameters."

Banneker did not record solutions to all these puzzles in his journal, but he certainly solved more complicated ones when he calculated by hand the exact times for eclipses. Imagine what he could have done with a calculator such as the TI-89!

Given that Banneker was self-taught and that he lived 200 years ago, his scientific and mathematical prowess are particularly remarkable. Although he was the son of a former slave and a farmer of modest means, he lived a life of unusual achievement. During his lifetime, he was often referred to as proof that African Americans were not intellectually inferior to European Americans. In 1791, Thomas Jefferson wrote Banneker:

> No body wishes more than I do to see such proofs as you exhibit, that nature has given to our black brethren, talents equal to those of the other colors of men, and that the appearance of a want of them is owing merely to the degraded condition of their existence, both in Africa & America. I can add with truth, that no body wishes more ardently to see a good system commenced for raising the condition both of their body & mind to what it ought to be, as fast as the imbecility of their present existence, and other circumstances which cannot be neglected, will admit . . ." (Bedini 1999, 164–65).

It should come as no surprise then that in 1980 the U.S. Postal Service issued a postage stamp in Banneker's honor.

REFERENCE

Bedini, Silvio A. *The Life of Benjamin Banneker: The First African American Man of Science.* 2nd ed. Baltimore, Md.: Maryland Historical Society, 1999.

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Solving Simultaneous Equations and Unit Conversion: A Guide

The Texas Instruments TI-89 is one of a number of calculators that contain a computer algebra system (CAS). Users of other CAS-capable calculators or computer programs should be able to easily adapt the commands used in this article to their equipment. The following is a guide to two of the TI-89 commands used in the article.

Solving simultaneous equations

Press F2 to access the Algebra menu, as shown in **figure 17,** and choose 1: solve(. Use the word *and* between equations. You can type the word *and* or locate it in the catalog, shown in **figure 18,** and paste it. Place a comma after the last equation, and follow it with the variables to be solved in set brackets.

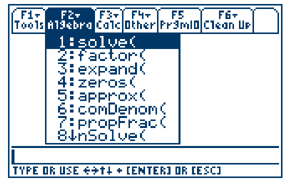

Fig. 17
The algebra menu of the TI-89

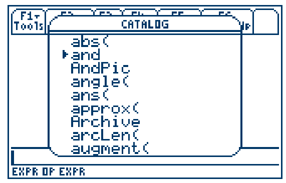

Fig. 18
Using the catalog to locate "and"

Unit conversion

To convert 39 imperial gallons to cubic inches, first enter 39, then press 2nd 3 to access the unit menu, as shown in **figure 19.**

As **figure 20** indicates, choose _galUK from the volume submenu. Press 2nd MODE to get the conversion symbol ▸, choose inch from the length submenu, and raise it to the third power by pressing ^ 3, as shown in **figures 21** and **22.** MT

Fig. 19

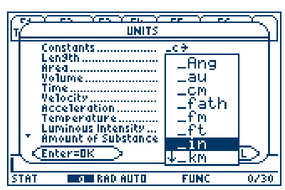

Fig. 21

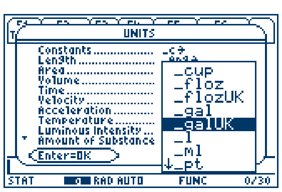

Fig. 20

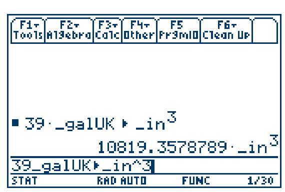

Fig. 22

Converting imperial gallons to cubic inches