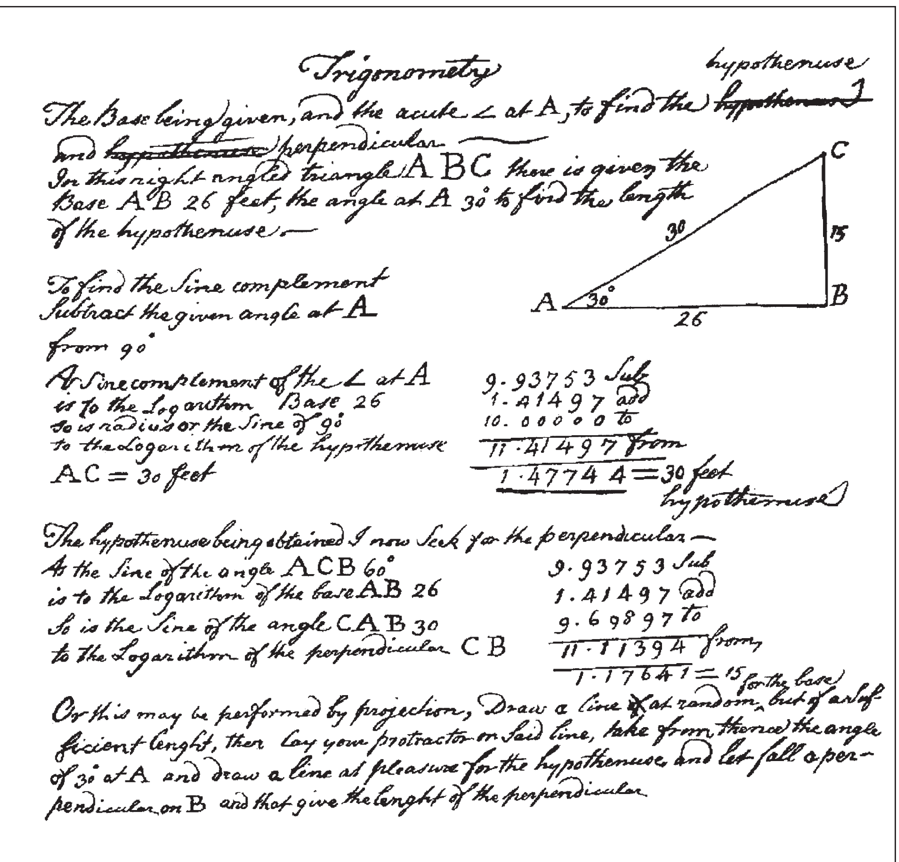
Benjamin Banneker and the Law of Sines

John F. Mahoney



Trigonometry

The Base being given, and the acute ∠ at A, to find the ~~hypothenuse~~ hypothenuse and ~~hypothenuse~~ perpendicular

In this right angled triangle ABC there is given the Base AB 26 feet, the angle at A 30° to find the length of the hypothenuse.—

To find the Sine complement
Subtract the given angle at A
from 90°

As Sine complement of the ∠ at A — 9.93753 Sub
is to the Logarithm Base 26 — 1.41497 add
So is radius or the Sine of 90° — 10.00000 to
to the Logarithm of the hypothenuse — 11.41497 from
AC = 30 feet — 1.47744 = 30 feet hypothenuse

The hypothenuse being obtained I now seek for the perpendicular —
As the Sine of the angle ACB 60° — 9.93753 Sub
is to the Logarithm of the base AB 26 — 1.41497 add
So is the Sine of the angle CAB 30 — 9.69897 to
to the Logarithm of the perpendicular CB — 11.11394 from
1.17641 = 15 for the base

Or this may be performed by projection, Draw a line at random, but of a sufficient lenght, then lay your protractor on said line, take from thence the angle of 30° at A and draw a line at pleasure for the hypothenuse, and let fall a perpendicular on B and that give the lenght of the perpendicular

Fig. 1 Page from Banneker's journal

Benjamin Banneker (1731–1806), the son of a freed slave from Guinea and a black woman, born free in the territory of Maryland, was a self-taught mathematician, surveyor, and astronomer. He lived and worked on the farm he owned with his father near Baltimore. His grandmother, an Englishwoman who had come to the colonies as an indentured servant, taught Banneker how to read and write. She arranged for him to go to a local one-room school, which was attended by several white and a few black children. The Quaker schoolmaster may have lent Banneker some books then, and later, a neighbor lent him some books on mathematics and astronomy. From these books he learned mathematics as far as double-position as well as surveying and astronomy. He began publishing annual almanacs containing his astronomical observations and predictions. He kept his notes in journals, but only one of the journals is preserved. This journal served as his notebook for astronomical observations, his diary, and his mathematics notebook. The mathematics in his journal consisted of six puzzles and two pages of mathematical writing.

With the aid of computer enhancement techniques we were able to reproduce Banneker's actual handwriting of the page shown in **figure 1.** His notes show both the mathematics he studied and the techniques he used. Teachers often encourage students to record their mathematical thinking in notebooks, much as Banneker did. His handwriting is transcribed and lines numbered in **figure 2.**

I used Banneker's work in my precalculus class after we had completed a unit on logarithms and a unit on the laws of sines and cosines. This work helped my students review these concepts and gave them insight into Banneker's own study of mathematics. Here is a lesson plan for the activity:

1. Photocopy Banneker's writing (**fig. 1**) onto an overhead transparency. Enlarging the figure 150 percent will make a nice transparency. Cover up the numbers 30 (the measure of side $\overline{AC}$) and 15 (the measure of side $\overline{BC}$) with opaque tape, and then project the top third (lines 1–8) of the page onto a screen.
2. Have a student in the class read aloud what Banneker wrote.
3. Ask the students how they would compute the lengths of sides $\overline{AC}$ and $\overline{BC}$. Some students might use the relationship $\cos 30° = 26/AC$ or $\tan 30° = BC/26$. Others might use the properties of a 30°-60°-90° triangle to compute the lengths. Either method gives the result of $AC = 30.0222$ and $BC = 15.0111$. After the students compute the lengths of $\overline{AC}$ and $\overline{BC}$, remove the opaque tape (from step 1) and verify that Banneker arrived at the same conclusions.
4. Are there other methods that would also work? With luck (and, maybe, some prodding), a student will suggest using the Law of Sines. Write the equation

$$\frac{\sin 60°}{26} = \frac{\sin 90°}{AC}$$

on the board and ask the students to solve it for AC:

$$\frac{0.866025}{26} = \frac{1}{AC} \Rightarrow AC = \frac{1 \bullet 26}{0.866025}$$

5. Have the students use logarithms to evaluate the last expression by first taking the log (base 10) of both sides of the equation:

$$\begin{aligned}\log AC &= \log \frac{1 \bullet 26}{0.866025} \\ &= \log 1 + \log 26 - \log 0.866025 \\ &= 0 + 1.41497 - (-0.06247)\end{aligned}$$

So $\log AC = 1.4774$ and thus $AC = 30.022$.
6. To show students how Banneker solved this problem more than 200 years ago without the aid of a calculator, reveal lines 9–13.
7. Have another student read what Banneker wrote to the left of his calculations.
8. Ask the students, "What is Banneker doing here?" Answer: He is using logarithms to apply the Law of Sines to solve this triangle as in step 4 above. To discuss why his calculations are not the same as the class calculations, inform the students that before calculators were readily available (in the early 1970s), students—and their teachers—had to use tables to find logarithms. To ensure that the values in these tables were positive (and to save the expense of printing additional tables), the logarithms of trigonometric functions were increased by 10. In line 9, Banneker indicated that he was finding the sine of the complement of A. In other words, he was

Trigonometry

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2. And perpendicular
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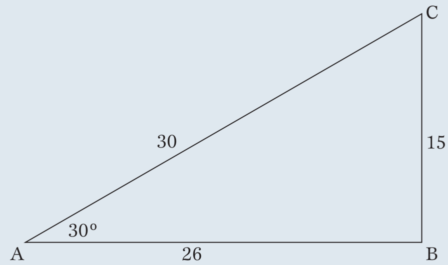

6. To find the Sine complement
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8. From 90°

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13. AC = 30 feet	1.47744 = 30 feet hypotenuse

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17. So is the Sine of the angle CAB 30	9.69897 to
18. to the Logarithm of the perpendicular CB	11.11394 from
19.	1.17641 = 15

20. Or this may be performed by projection, Draw a line at random for the base but of a Suf-
21. ficient lenght, then lay your protractor on Said line, take from thence the angle
22. of 30° at A and draw a line at pleasure for the hypotenuse and let fall a per-
23. pendicular on B and that give the lenght of the perpendicular.

Fig. 2 Page from figure 1, transcribed

*42 Log. Sines. Deg. 53, 54, 55, 56, 57, 58, 59, 60. In. 9.										
D '	0	1	2	3	4	5	6	7	8	9
53.0	902.349	444	539	634	729	824	919	014	108	203
10	903.298	392	487	581	676	770	864	959	053	147
20	904.241	335	429	523	617	711	804	898	991	085
30	905.179	272	366	459	552	645	739	832	925	017
40	906.111	204	296	389	482	574	667	760	852	945
50	907.037	129	222	314	406	498	591	682	774	866
54.0	958	049	141	233	324	416	507	599	690	781
10	908.873	964	055	146	237	328	419	510	601	691
20	909.782	873	963	054	144	235	325	415	506	596
30	910.686	776	866	956	046	136	226	315	405	495
40	911.584	674	763	853	942	031	121	210	299	388
50	912.477	566	655	744	833	921	010	099	187	276
55.0	913.364	453	541	630	718	806	894	982	070	158
10	914.246	334	422	510	598	685	773	860	948	035
20	915.123	210	297	385	472	559	646	733	820	907
30	994	080	167	254	341	427	514	600	687	773
40	916.859	945	032	118	204	290	376	462	548	634
50	917.719	805	891	976	062	147	233	318	404	489
56.0	918.574	659	744	830	915	000	084	169	254	339
10	919.424	508	593	677	762	846	931	015	099	184
20	920.268	352	436	520	604	688	772	855	939	023
30	921.107	190	174	357	441	524	607	691	774	857
40	941	023	106	189	272	355	438	520	603	685
50	922.768	851	933	016	098	180	263	345	427	509
57.0	923.591	673	755	837	919	001	083	164	246	328
10	924.409	491	572	653	735	816	897	978	060	141
20	925.222	303	384	465	545	626	707	787	868	949
30	926.029	110	190	270	351	431	511	591	671	751
40	831	911	991	071	151	231	310	390	469	549
50	927.628	708	787	867	946	025	104	183	262	341
58.0	928.420	499	578	657	736	814	893	972	050	129
10	929.207	286	364	442	521	599	677	755	833	911
20	989	067	145	223	300	378	456	533	611	688
30	930.766	843	920	998	075	152	229	306	383	460
40	931.537	611	691	768	845	921	998	074	151	227
50	932.304	380	457	533	609	685	762	838	914	990
59.0	933.066	141	217	293	369	444	520	596	671	747
10	822	897	973	048	123	198	274	349	424	499
20	934.574	649	723	798	873	948	022	097	171	246
30	935.320	395	469	543	618	692	766	840	914	988
40	936.062	136	210	283	357	431	505	578	652	725
50	799	872	945	019	092	165	238	311	385	458
60.0	937.531	603	674	749	822	895	967	040	113	185
10	938.258	330	402	475	547	619	691	763	835	908
20	979	051	123	195	267	339	410	482	554	625
30	939.697	768	839	911	982	053	125	196	267	338
40	940.409	480	515	622	693	763	834	905	975	046
50	941:117	187	257	328	398	468	539	609	679	749

Fig. 3 Table of Logarithms of Sines from Benjamin Martin, *Logarithmologia: Or the Whole Doctrine of Logarithms*. London: Printed for J. Hodges, at the Looking Glass on London Bridge, 1740.

Banneker's computations	Annotations
9.93753 Sub 1.41497 add 10.00000 to 11.41497 from 1.47744 = 30 feet hypotenuse	Banneker looks up the log of sin 60° in a log trig table. He looks up the log of 26 in a log table and he adds that value to the log of sin 90° from the log trig table. From this sum he subtracts the log of sin 60° to get the log of the length of the hypotenuse. Finally he takes the antilog of the result to get *AC*. Note that the 10s added to log sin 60° and log sin 90° cancel when subtracted.

Fig. 4 Banneker's computations

finding the sine of 60°, the angle at *C*. He then looked up the log sin 60° in a table of logarithms of trigonometric functions. His result was 9.93753, and he would have known that this really meant that log sin 60° = 9.93753 – 10 or –0.06247. An abbreviated log sine table from a 1740 textbook is shown in **figure 3.** The annotations in **figure 4** show that Banneker's computations are equivalent to steps 4 and 5 above.

9. Another question to ask the students is, Why didn't Banneker just divide 26 by 0.866025 to find *AC?* Division by decimals is hard and errors can easily occur. Logarithms were invented to make calculations easier and faster. Ask your students to verify this by having them perform this computation without the aid of a calculator.

   At this point your students should be challenged to calculate the length of $\overline{BC}$ by Banneker's method.
10. Now refer to Banneker's work on lines 14–19 and compare the results.

Banneker also gives a geometric method (lines 20–23) of constructing the lengths of the sides of the triangle.

Banneker's 15-26-30-sided triangle is an excellent integer-sided approximation to a 30°-60°-90° triangle. I have examined a number of eighteenth- and nineteenth-century American math textbooks and have not found any references to this approximation. Perhaps Banneker knew the dimensions of this triangle and used it to practice the Law of Sines, or perhaps he used this triangle while surveying to quickly get 30° or 60° angles.

As an assignment based on Banneker's work, ask the students to calculate the exact angle measurements for Banneker's 15-26-30-sided triangle. The solution, using the law of cosines:

$$\cos C = \frac{15^2 + 26^2 - 30^2}{2 \cdot 15 \cdot 26}$$

$$= 0.001282 \Rightarrow C = 89.92654°$$

Similarly, the other two angles are 29.99997° and 60.07348°.

I also used Banneker's work to illustrate the importance of maintaining an organized mathematical notebook. I asked my students, "Could someone else read your notebook and determine what you were doing? If not, how can you hope to use your notebook as a plan to study for exams at the end of the year?" Banneker's work also illustrates the fact that the method used to solve a mathematical problem depends on the tools available. Mathematics considered important to one generation—logarithms of trigonometric functions, in this example—may be nearly obsolete for another generation. This raises the question for both students and teachers: What mathematics is important to teach and to learn?

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Banneker's journal is part of the Schomburg Collections of the New York Public Library. It was converted to microfilm by 3M's International Microfilm Press, 1970. Many universities and historical societies have copies of the microfilm. I viewed it at the Maryland Historical Society in Baltimore. Unfortunately, the quality of the reproduction is quite poor. My son, Quinn, a student at MIT, scanned my photocopy of Banneker's work. I sought additional help through the international math history list server mathforum.org/epigone/historia_matematica. I was contacted by Omar Rumi, a retired mathematics teacher in Kuala Lumpur, Malaysia, who generously offered his time. Over a period of four months, Omar painstakingly cleaned up the copies pixel by pixel. I am indebted to him. These examples of Banneker's handwriting can be downloaded from web.mit.edu/qmahoney/www/nctm/. ∞

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